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Comparison between the rotating wave and Feynman-Vernon system-reservoir couplings in the non-Markovian regime

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Abstract. This paper deals with the dissipative dynamics of a quantum harmonic oscillator interacting with a bosonic reservoir. The Master Equations based on the Rotating Wave and on the Feynman-Vernon system-reservoir couplings are compared highlighting differences and analogies. We discuss quantitatively and qualitatively the conditions under which the counter rotating terms can be neglected. By comparing the analytic solution of the heating function relative to the two different coupling models we conclude that, even in the weak coupling limit, the counter rotating terms give rise to a significant contribution in the non-Markovian short time regime. The main result of this paper is that such a contribution is actually experimentally measurable and thus relevant for a correct description of the system dynamics.

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1 Introduction

During the last few decades a huge deal of attention has been devoted to the study of the quantum dynamics of dissipative systems. The theory of open quantum systems, indeed, is essential for the understanding of a variety of physical phenomena in different fields of physics, such as, for example, quantum optics and solid state physics [1]. Moreover, very recently, there has been an increasing interest in the effects of decoherence, due to the unavoidable coupling with external environment [2–4], on the dynamics of quasi-closed systems used for quantum computing and quantum information processing. The usual approach for studying decoherence and dissipation effects starts by prescribing a total Hamiltonian for the closed total system (system+reservoir). Then, after tracing over the reservoir variables and performing, if necessary, appropriate approximations, one finally derives a Master Equation ruling the dynamics of the dissipative quantum system. One of the most commonly done assumptions for describing open quantum systems is the so-called Born-Markov approximation which basically consists in neglecting memory effects of the reservoir. In other words one assumes that the correlation time of the reservoir, characterizing the time scale on which the reservoir memory would feed back to the system, is much shorter than the typical relaxation time of the system. When such condition is satisfied it is possible to derive a Master Equation describing the time evolution of the dissipative system for times longer than the correlation time of the reservoir. Under this approximation the resulting Master Equation is called Markovian Master Equations and of course does not describe appropriately systems interacting with natural or engineered structured reservoirs, such as atoms decaying in photonic band gap materials or atom lasers.

The approach reported in this paper does not rely on the Born-Markov approximation. The only basic assumption of the method used is weak coupling between system and reservoir. For this reason our approach allows to describe non-Markovian system behaviors.

It has been very recently demonstrated by Ahn *et al.* [5] that non-Markovian reservoirs may be of potential interest for quantum information processing since a quantum system is decohered slower in a non-Markovian reservoir than in a Markovian one.

Recently a huge deal of attention has been devoted to the short time evolution of quantum mechanical systems in connection to Quantum Zeno Effect (QZE) [6]. A typical feature of the Quantum Zeno dynamics is the quadratic behavior at short times of the survival probability of the initial state of the system, due to persistence of quantum phase correlations. In this sense the QZE is a manifestation of the deviation from Markovian dynamics at short times. The mathematical approach we use in this paper allows to study the non-Markovian short time regimes and therefore it is appropriate for describing Quantum Zeno phenomena.

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In this paper we firstly derive the time-convolutionless Master Equation describing a quantum harmonic oscillator of frequency ω_0 interacting with a bosonic reservoir represented as an infinite chain of harmonic oscillators of frequencies $\{\omega_i\}$ [9–11]. The method used, exploiting a superoperatorial formalism, leads to a non-Markovian Master Equation spoiled of reservoir memory kernels [7,8]. In words one says that such a Master equation is local in time. By definition this means that such an equation of motion for the density matrix is characterized by the fact that the time derivative of $\hat{\rho}(t)$ only depends on the actual value of $\hat{\rho}(t)$. On the contrary, in non-Markovian Master Equations involving memory kernels, the time derivative of $\hat{\rho}(t)$ is related to values $\hat{\rho}(s)$ of the density matrix at times $s < t$.

Our aim is to analyze differences and analogies in the dynamical behaviour of this specific open system in correspondence with two different prefixed system-reservoir couplings. The first choice is the following:

$$\hat{H}_{sr}^{\mathrm{RWA}} = \alpha \sum_{i=0}^{\infty} \hbar \sqrt{\frac{\omega_i}{2}} \left(g_i \hat{a} \hat{b}_i^{\dagger} + \mathrm{h.c.} \right) \equiv \alpha \left(\hat{a} \hat{R}^{\dagger} + \hat{a}^{\dagger} \hat{R} \right), \tag{1}$$

usually referred to as Rotating Wave (RW) coupling. In equation (1), $\hat{a}$ and $\hat{b}_i$ are the annihilation operators of the system and reservoir harmonic oscillators respectively and α is the adimensional coupling constant. Note that, for the sake of simplicity, in the paper we use adimensional position and momentum operators for the system oscillator.

The second form of the system-reservoir interaction Hamiltonian examined in this paper is the so-called Feynman-Vernon (FV) coupling [12]:

$$\hat{H}_{sr} = \alpha \hat{X} \sum_{i=0}^{\infty} \hbar \sqrt{\omega_i} \left(g_i^* \hat{b}_i + g_i \hat{b}_i^{\dagger} \right) \equiv \alpha \hat{X} \left(\hat{R} + \hat{R}^{\dagger} \right), \tag{2}$$

where the operator $\hat{X}$ is related to the creation and annihilation operators of the quantum harmonic oscillator simply as

$$\hat{X} = \frac{1}{\sqrt{2}} \left(\hat{a} + \hat{a}^{\dagger} \right). \tag{3}$$

While the first interaction Hamiltonian is very often used in describing quantum optics systems [13] and atom lasers [14], the second one leads to the Master Equation for Quantum Brownian Motion [15]. Using the Hamiltonian given by equation (1) instead of the more general one given by equation (2) is usually motivated saying that the counter rotating terms $\hat{a}\hat{b}_j$ and $\hat{a}^{\dagger}\hat{b}_j^{\dagger}$, appearing in equation (2), do not conserve the total unperturbed energy and thus give a negligible contribution to the system dynamics in the weak coupling limit.

The main result of this paper is that, in the non-Markovian regime, the contribution given by the counter-rotating terms is not negligible and experimentally measurable, also when the weak coupling limit is invoked.

The paper is structured as follows. In Section 2 we introduce the superoperator formalism for the derivation of non-Markovian generalized Master Equations. In Section 3 we specialize the generalized Master Equations to the cases of Rotating Wave and Feynman-Vernon couplings and we compare them in Section 4. Finally in Section 5 we present conclusions.

2 Derivation of the Master Equation: an operatorial approach

Let us consider an open quantum system interacting with an environment whose physical nature needs not to be specified at this moment. We indicate the total Hamiltonian as follows

$$\hat{H} = \hat{H}_0 + \hat{H}_E + \alpha \hat{H}_{\mathrm{int}}, \tag{4}$$

where $\hat{H}_0$, $\hat{H}_E$ and $\hat{H}_{\mathrm{int}}$ stand for the system, environment and interaction Hamiltonians respectively and α is the coupling constant. The Von Neumann-Liouville equation for the total system, in the interaction picture, is the following

$$\frac{\mathrm{d}\tilde{\rho}(t)}{\mathrm{d}t} = \frac{\alpha}{\mathrm{i}\hbar} \left[\hat{H}_I(t), \tilde{\rho}(t) \right] \equiv \frac{\alpha}{\mathrm{i}\hbar} \mathbf{H}_I^S(t) \tilde{\rho}(t). \tag{5}$$

In equation (5), $\tilde{\rho}$ and $\hat{H}_I(t)$ are the density matrix and the interaction Hamiltonian of the total system respectively, in the interaction picture, and the superoperator $\mathbf{H}_I^S(t)$ is defined as $\mathbf{H}_I^S(t) = [\hat{H}_I(t), \cdot\ ]$. In the rest of the paper, given a certain operator $\hat{A}$, we will use the following notation for the “commutator ”and “anticommutator”superoperators:

$$\mathbf{A}^S = [\hat{A}, \cdot\ ] \qquad \mathbf{A}^{\Sigma} = \left\{ \hat{A}, \cdot\ \right\} \cdot \tag{6}$$

In deriving the generalized Master Equation we assume that at $t = 0$ system and environment are uncorrelated, that is $\tilde{\rho}(0) = \hat{\rho}(0) \bigotimes \hat{\rho}_E(0)$, with $\hat{\rho}$ and $\hat{\rho}_E$ density matrices of system and environment respectively and that the environment is stationary, *i.e.* $[\hat{H}_E, \hat{\rho}_E] = 0$.

A formal solution of equation (5) can be written as

$$\tilde{\rho}(t) = \mathbf{T}(t) \tilde{\rho}(0), \tag{7}$$

where the superoperator $\mathbf{T}(t)$ is defined as the solution of the equation:

$$\dot{\mathbf{T}}(t) = \frac{\alpha}{\mathrm{i}\hbar} \mathbf{H}_I^S(t) \mathbf{T}(t), \tag{8}$$

with $\mathbf{T}(0) = \mathbf{1}$. Remembering that $\hat{\rho}(t) = \mathrm{Tr}_E \{\tilde{\rho}(t)\}$ and that $\tilde{\rho}(0) = \hat{\rho}(0) \bigotimes \hat{\rho}_E(0)$, after tracing over the environmental variables, equation (7) becomes

$$\begin{aligned} \hat{\rho}(t) &= \mathrm{Tr}_E \{\mathbf{T}(t) \hat{\rho}_E(0)\} \hat{\rho}(0) \\ &\equiv \langle \mathbf{T}(t) \rangle \hat{\rho}(0) \equiv (1 + \mathbf{M}(t)) \hat{\rho}(0), \end{aligned} \tag{9}$$

where we have indicated with $\langle \mathbf{T}(t) \rangle$ the superoperator $(1 + \mathbf{M}(t)) = \mathrm{Tr}_E \{\mathbf{T}(t)\hat{\rho}_E(0)\}$, acting on the space $\mathcal{H}_s \bigotimes \mathcal{H}_s^*$, with $\mathcal{H}_s$ Hilbert space of the system. Differentiating now equation (9) yields

$$\frac{\mathrm{d}\hat{\rho}(t)}{\mathrm{d}t} = \dot{\mathbf{M}}(t)\hat{\rho}(0). \tag{10}$$

Inserting in equation (10) the expression for $\hat{\rho}(0)$ obtained inverting equation (9) gives

$$\frac{\mathrm{d}\hat{\rho}(t)}{\mathrm{d}t} = \left(\dot{\mathbf{M}}(t)\right)[1 + \mathbf{M}(t)]^{-1}\hat{\rho}(t) \equiv \mathbf{K}(t)\hat{\rho}(t). \tag{11}$$

In the previous equation we have defined a new superoperator $\mathbf{K}(t)$ acting on the space $\mathcal{H}_s \bigotimes \mathcal{H}_s^*$ too. At this point it is worth spending few words on the existence of $\mathbf{K}(t)$, that is of the inverse superoperator $[1 + \mathbf{M}(t)]^{-1}$. To this aim, we recast $\mathbf{K}(t)$ in the form

$$\mathbf{K}(t) = \left(\dot{\mathbf{M}}(t)\right)[1 + \mathbf{M}(t)]^{-1} = \left(\dot{\mathbf{M}}(t)\right)\sum_n (-\mathbf{M}(t))^n . \tag{12}$$

The problem of the invertibility of the superoperator $[1 + \mathbf{M}(t)]$ has been partly addressed in references [16,8]. Although, at the best of our knowledge, a rigorous mathematical study of the conditions under which such a superoperator can be inverted does not exist in the literature, the following considerations can be done. The superoperator $\mathbf{M}(t)$ has the obvious properties $\mathbf{M}(0) = 0$ and $\mathbf{M}(t)|_{\alpha=0} = 0$, as one can directly infer from equations (8) and (9). Hence, invoking the continuity of $\mathbf{M}(t)$ in $t = 0$, $[1 + \mathbf{M}(t)]$ may be inverted for not too large couplings and small t. For large couplings and/or large time intervals it may happen that the inverse of such an operator does not exist. It is worth noting, however, that time-convolutionless Master Equations have been successfully used to describe non Markovian dynamics in many different physical situations (see Chapt. 10 of Ref. [8] for a review).

Now, it is easy to convince oneself that a formal solution of equation (8) may be written as

$$\mathbf{T}(t) = \exp_c\left[\frac{\alpha}{\mathrm{i}\hbar}\int_0^t \mathbf{H}_I^S(t_1)\,\mathrm{d}t_1\right] \equiv \sum_{n=0}^{\infty}\left(\frac{\alpha}{\mathrm{i}\hbar}\right)^n \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} \mathbf{H}_I^S(t)\mathbf{H}_I^S(t_1)\cdots\mathbf{H}_I^S(t_n)\,\mathrm{d}t_1\cdots\mathrm{d}t_n, \tag{13}$$

where the subscript c in the exponential stands for the Dyson chronological order, *i.e.* $t_n > t_{n-1} \cdots > t_1 > t$. Inserting such an expression into equation (11) with the help of equation (9) and collecting all the terms proportional to the same power in α, it is possible to demonstrate that the following expansion holds:

$$\mathbf{K}(t) = \sum_{n=0}^{\infty} \mathbf{k}_n(t) \tag{14}$$

with

$$\mathbf{k}_n(t) = \left(\frac{\alpha}{\mathrm{i}\hbar}\right)^n \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} \langle\langle \mathbf{H}_I^S(t)\mathbf{H}_I^S(t_1)\cdots\mathbf{H}_I^S(t_n)\rangle\rangle_{o.c.}\,\mathrm{d}t_1\cdots\mathrm{d}t_n. \tag{15}$$

In the previous equation we have indicated with $\langle\langle\cdots\rangle\rangle_{o.c.}$ the temporal ordered cumulants [19]. As an example, we report the expression of the first and second cumulants, respectively:

$$\begin{aligned}\langle\langle \mathbf{H}_I^S(t)\rangle\rangle_{o.c.} &= \langle \mathbf{H}_I^S(t)\rangle \\ \langle\langle \mathbf{H}_I^S(t)\mathbf{H}_I^S(t_1)\rangle\rangle_{o.c.} &= \langle \mathbf{H}_I^S(t)\mathbf{H}_I^S(t_1)\rangle \\ &\quad - \langle \mathbf{H}_I^S(t)\rangle\langle \mathbf{H}_I^S(t_1)\rangle.\end{aligned} \tag{16}$$

The form of equation (15) resembles the result obtained by Van Kampen in the context of stochastic differential equations [17,18].

The origin of the expansion given by equations (14) and (15) can be understood as follows. Let us write the superoperator $\mathbf{K}(t)$ defined by equation (12) in following symbolic form

$$\mathbf{K}(t) = \frac{\delta}{\delta t}\ln\left[\left\langle \exp_c\left(\frac{\alpha}{\mathrm{i}\hbar}\int_0^t \mathbf{H}_I^S(t_1)\,\mathrm{d}t_1\right)\right\rangle\right], \tag{17}$$

where

$$\frac{\delta}{\delta t}F[\mathbf{A}(t)] \equiv \left(\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{A}(t)\right)F'[\mathbf{A}]. \tag{18}$$

In equation (17) the symbol $\langle\cdots\rangle = \mathrm{Tr}_E\{\cdots\hat{\rho}_E\}$ describes an operation of average over the environmental degrees of freedom. The expression $\langle \exp_c(\frac{\alpha}{\mathrm{i}\hbar}\int_0^t \mathbf{H}_I^S(t_1)\,\mathrm{d}t_1))\rangle$ can thus be seen as the generalization, in the superoperator formalism, of the concept of characteristic functional [20]. As a consequence the superoperator $\ln[\langle \exp_c(\frac{\alpha}{\mathrm{i}\hbar}\int_0^t \mathbf{H}_I^S(t))\rangle]$ is the generalization of the generator of cumulants introduced in standard textbooks. This circumstance makes it clear why the integrand in equation (15) is called temporal ordered cumulant. In view of equation (17) the existence problem of the superoperator $\mathbf{K}(t)$ can be traced back to the convergence of the series of cumulants in equation (14).

In order to derive the explicit form of the generalized Master Equation, let us assume a bilinear interaction Hamiltonian of the form:

$$\hat{H}_I(t) = \alpha\sum_{i=1}^n \hat{A}_i(t)\hat{E}_i(t) = \alpha\hat{\boldsymbol{A}}(t)\cdot\hat{\boldsymbol{E}}(t), \tag{19}$$

where $\hat{\boldsymbol{A}}(t) \equiv \left(\hat{A}_1(t), \hat{A}_2(t), \ldots, \hat{A}_n(t)\right)$ and $\hat{\boldsymbol{E}}(t) \equiv \left(\hat{E}_1(t), \hat{E}_2(t), \ldots, \hat{E}_n(t)\right)$ are system and environment operators respectively. In the weak coupling limit we may stop the cumulant expansion given in equation (14) to the second order in the coupling constant. In view of equations (11, 14) and (15), this leads to the following Master Equation

$$\frac{\mathrm{d}\hat{\rho}(t)}{\mathrm{d}t} = \frac{\alpha}{\mathrm{i}\hbar}\langle \mathbf{H}_I^S(t)\rangle\hat{\rho}(t) - \frac{\alpha^2}{\hbar^2}\int_0^t \left[\langle \mathbf{H}_I^S(t)\mathbf{H}_I^S(t_1)\rangle - \langle \mathbf{H}_I^S(t)\rangle\langle \mathbf{H}_I^S(t_1)\rangle\right]\mathrm{d}t_1\hat{\rho}(t). \tag{20}$$

Assuming for simplicity that the form of the environmental density matrix satisfy the condition $\langle \hat{E}_i\rangle = \mathrm{tr}\hat{E}_i\hat{\rho}_E E = 0$ (as for example in the case of a thermal reservoir), one can show that the first term of equation (20) vanishes at every time t. The explicit manipulation of the second term is presented in Appendix A and leads to the following final form of the non-Markovian generalized Master Equation:

$$\begin{aligned}\frac{\mathrm{d}\hat{\rho}(t)}{\mathrm{d}t} &= -\sum_{i,\,j=1}^{n}\left(\int_0^t \left[\kappa_{i,j}(\tau)\mathbf{A}_i^S(t)\mathbf{A}_j^S(t-\tau)\right.\right.\\ &\quad \left.\left. - \mathrm{i}\mu_{i,j}(\tau)\mathbf{A}_i^S(t)\mathbf{A}_j^{\Sigma}(t-\tau)\right]\mathrm{d}\tau\right)\hat{\rho}(t)\\ &\equiv -\left[\mathbf{D}(t) - \imath\mathbf{G}(t)\right]\hat{\rho}(t) \equiv \mathbf{L}(t)\hat{\rho}(t),\end{aligned} \tag{21}$$

where definitions of equation (6) have been used. In equation (21) we have introduced the environment correlation $\kappa_{i,j}(\tau)$ and susceptibility $\mu_{i,j}(\tau)$ matrices, with $\tau = t - t_1$. Such quantities, characterizing the temporal behavior of the environment, are defined as follows

$$\kappa_{i,j}(\tau) = \frac{\alpha^2}{2\hbar^2}\left\langle\left\{\hat{E}_i(\tau), \hat{E}_j(0)\right\}\right\rangle, \tag{22}$$

$$\mu_{i,j}(\tau) = \frac{\mathrm{i}\alpha^2}{2\hbar^2}\left\langle\left[\hat{E}_{\mathrm{i}}(\tau), \hat{E}_j(0)\right]\right\rangle\cdot \tag{23}$$

The form of equation (21) has a clear physical meaning. One can show that the superoperator $\mathbf{D}(t)$ is strictly connected with diffusion (decoherence) processes only [21]. The superoperator $\mathbf{G}(t)$, describing the dissipation processes and frequency renormalization, on the other hand, arises from a quantum mechanical treatment of the environment and, indeed, vanishes when a semiclassical description of the environment is used (see also Eq. (23)) [21].

In the next section we further carry on the calculations in order to obtain and compare the two non-Markovian Master Equations corresponding to the Rotating Wave and Feynman-Vernon couplings respectively.

3 Rotating wave and Feynman-Vernon couplings in the non-Markovian regime

Let us consider a quantum harmonic oscillator whose Hamiltonian is given by:

$$\hat{H}_0 = \frac{\hbar\omega_0}{2}\left(\hat{P}^2 + \hat{X}^2\right) = \hbar\omega_0\left(\hat{a}^\dagger\hat{a} + 1/2\right), \tag{24}$$

with ω_0 frequency of the harmonic oscillator. The system interacts with a bosonic reservoir at T temperature of Hamiltonian

$$\hat{H}_E = \hbar\sum_{i=0}^{\infty}\omega_i\left(\hat{b}_i^\dagger\hat{b}_i + 1/2\right), \tag{25}$$

with ω_i frequencies of the reservoir oscillators.

3.1 Feynman-Vernon coupling

Let us begin discussing the Feynman-Vernon interaction Hamiltonian, given by equation (2). Such a coupling is of the form of equation (19) where, in the interaction picture, $\hat{A}(t) = \hat{X}(t)$ and $\hat{E}(t) = \hat{R}(t) + \hat{R}^\dagger(t)$. Our aim is to manipulate equation (21) in order to obtain the specific non-Markovian generalized Master Equation appropriate for our system. In this section we will sketch the main steps of the derivation. More details can be found in Appendix B.

First of all let us write the Master Equation given in equation (21) in the Schrödinger picture. Introducing the superoperator

$$\mathbf{T}_0(t) = \exp\left[\frac{1}{\mathrm{i}\hbar}\mathbf{H}_0^S t\right], \tag{26}$$

with $\hat{H}_0$ given by equation (24), and transforming in the Schrödinger picture the superoperator $\mathbf{L}(t)$ defined in equation (21)

$$\mathbf{L}_S(t) = \mathbf{T}_0(t)\,\mathbf{L}(t)\,\mathbf{T}_0^{-1}(t), \tag{27}$$

our generalized Master Equation becomes

$$\frac{\mathrm{d}\hat{\rho}_S(t)}{\mathrm{d}t} = \left[\frac{1}{\mathrm{i}\hbar}\mathbf{H}_0^S - \mathbf{D}_S(t) + \mathrm{i}\mathbf{G}_S(t)\right]\hat{\rho}_S(t), \tag{28}$$

with $\hat{\rho}_S$ density matrix of the harmonic oscillator in the Schrödinger picture. In Appendix B we show that the superoperators $\mathbf{D}_S(t)$ and $\mathbf{G}_S(t)$ can be recast in the form

$$\mathbf{D}_S(t) = \int_0^t \kappa(\tau)\mathbf{X}^S\left(\cos\omega_0\tau\mathbf{X}^S - \sin\omega_0\tau\mathbf{P}^S\right)\mathrm{d}\tau, \tag{29}$$

$$\mathbf{G}_S(t) = \int_0^t \mu(\tau)\mathbf{X}^S\left(\cos\omega_0\tau\mathbf{X}^{\Sigma} - \sin\omega_0\tau\mathbf{P}^{\Sigma}\right)\mathrm{d}\tau, \tag{30}$$

where $\mathbf{P}^S$ and $\mathbf{P}^{\Sigma}$ are the "commutator" and "anticommutator" superoperators associated to the operator

$$\hat{P} = \frac{\mathrm{i}}{\sqrt{2}}\left(\hat{a}^\dagger - \hat{a}\right), \tag{31}$$

and $\kappa(\tau) \equiv \kappa_{1,1}(\tau)$ and $\mu(\tau) \equiv \mu_{1,1}(\tau)$ are defined by equations (22–23). Inserting equations (29–30) into equation (28) we get

$$\begin{aligned}\frac{\mathrm{d}\hat{\rho}_S(t)}{\mathrm{d}t} &= \frac{1}{\mathrm{i}\hbar}\mathbf{H}_0^S - \Big[\bar{\Delta}(t)(\mathbf{X}^S)^2 - \Pi(t)\mathbf{X}^S\mathbf{P}^S\\ &\quad -\frac{\mathrm{i}}{2}r(t)(\mathbf{X^2})^S + \mathrm{i}\gamma(t)\mathbf{X}^S\mathbf{P}^{\Sigma}\Big]\hat{\rho}_S(t).\end{aligned} \tag{32}$$

The time dependent coefficients appearing in the previous equation are defined as follows

$$\bar{\Delta}(t) = \int_0^t \kappa(\tau)\cos(\omega_0\tau)\mathrm{d}\tau, \quad (33)$$

$$\gamma(t) = \int_0^t \mu(\tau)\sin(\omega_0\tau)\mathrm{d}\tau, \quad (34)$$

$$\Pi(t) = \int_0^t \kappa(\tau)\sin(\omega_0\tau)\mathrm{d}\tau, \quad (35)$$

$$r(t) = 2\int_0^t \mu(\tau)\cos(\omega_0\tau)\mathrm{d}\tau. \quad (36)$$

From the form of equation (32), and remembering that $\mathbf{H}_0^S$ may be written as

$$\mathbf{H}_0^S = \frac{1}{2}\left[(\mathbf{P^2})^S + (\mathbf{X^2})^S\right], \quad (37)$$

it is not difficult to convince oneself that the term having coefficient $r(t)$ gives a renormalization of the oscillator frequency.

As usually done in standard textbooks [13], this term can be included in the definition of ω_0. In the following we neglect such term since it is possible to prove that such an approximation is always justified in the weak coupling regime $\alpha \ll 1$, provided that the reservoir frequency cut-off remains finite.

Under these conditions, the Master Equation, in the interaction picture with respect to $\hat{H}_0$, assumes the form

$$\frac{\mathrm{d}\hat{\rho}(t)}{\mathrm{d}t} = -\left[\bar{\Delta}(t)(\mathbf{X}^S)^2(t) - \Pi(t)\mathbf{X}^S(t)\mathbf{P}^S(t) + \mathrm{i}\gamma(t)\mathbf{X}^S(t)\mathbf{P}^\Sigma(t)\right]\hat{\rho}(t). \quad (38)$$

The time dependent superoperators appearing in equation (38) are those related to the operators

$$\hat{X}(t) = \hat{X}\cos(\omega_0 t) + \hat{P}\sin(\omega_0 t), \quad (39)$$

$$\hat{P}(t) = \hat{P}\cos(\omega_0 t) - \hat{X}\sin(\omega_0 t). \quad (40)$$

Equation (38) can be exactly solved in an operatorial way and the solution has an operatorial form [23,24]. This fact may be exploited to fully disclose both the short time non-Markovian and the asymptotic Markovian behaviors characterizing the dynamics of the system, as we will see in Section 4.

3.2 Rotating wave coupling

The generalized Master Equation correspondent to the interaction Hamiltonian given by equation (1), derived following the same procedure presented in the previous section, is (see also Appendix C)

$$\frac{\mathrm{d}\hat{\rho}(t)}{dt} = \left[-\bar{\Delta}^{\mathrm{RWA}}(t)\mathbf{a}^{\dagger S}\mathbf{a}^S - \frac{\gamma^{\mathrm{RWA}}(t)}{2}\left(\mathbf{a}^{\dagger S}\mathbf{a}^\Sigma - \mathbf{a}^S\mathbf{a}^{\dagger\Sigma}\right) + \frac{\mathrm{i}}{2}r^{\mathrm{RWA}}(t)\left(\mathbf{a}^{\dagger S}\mathbf{a}^\Sigma + \mathbf{a}^S\mathbf{a}^{\dagger\Sigma}\right)\right]\hat{\rho}(t). \quad (41)$$

The time dependent coefficients appearing in this equation are defined as follows

$$\bar{\Delta}^{\mathrm{RWA}}(t) = \int_0^t \kappa^{\mathrm{RWA}}(\tau)\mathrm{d}\tau, \quad (42)$$

$$\gamma^{\mathrm{RWA}}(t) = \int_0^t \mu_R^{\mathrm{RWA}}(\tau)\mathrm{d}\tau, \quad (43)$$

$$r^{\mathrm{RWA}}(t) = 2\int_0^t \mu_I^{\mathrm{RWA}}(\tau)\mathrm{d}\tau, \quad (44)$$

where

$$\kappa^{\mathrm{RWA}}(\tau) = \frac{\alpha^2}{2\hbar^2}\left\langle\left\{\hat{\tilde{R}}(\tau), \hat{\tilde{R}}^\dagger(0)\right\} + \left\{\hat{\tilde{R}}^\dagger(\tau), \hat{\tilde{R}}(0)\right\}\right\rangle, \quad (45)$$

$$\mu_R^{\mathrm{RWA}}(\tau) = \frac{\mathrm{i}\alpha^2}{2\hbar^2}\left\langle\left[\hat{\tilde{R}}(\tau), \hat{\tilde{R}}^\dagger(0)\right] + \left[\hat{\tilde{R}}^\dagger(\tau), \hat{\tilde{R}}(0)\right]\right\rangle, \quad (46)$$

$$\mu_I^{\mathrm{RWA}}(\tau) = \frac{\mathrm{i}\alpha^2}{2\hbar^2}\left\langle\left[\hat{\tilde{R}}(\tau), \hat{\tilde{R}}^\dagger(0)\right] - \left[\hat{\tilde{R}}^\dagger(\tau), \hat{\tilde{R}}(0)\right]\right\rangle, \quad (47)$$

and $\hat{\tilde{R}}(t)$ is the operator (see Appendix C)

$$\hat{\tilde{R}}(t) \equiv \sum_{i=0}^{\infty}\hbar\sqrt{\frac{\omega_i}{2}}g_i\hat{b}_i\mathrm{e}^{-\mathrm{i}(\omega_i-\omega_0)t}. \quad (48)$$

In the Schrödinger picture equation (41) takes the form

$$\frac{\mathrm{d}\hat{\rho}_S(t)}{\mathrm{d}t} = \left[\frac{1}{\mathrm{i}\hbar}\mathbf{H_0}^S + \mathrm{i}r^{\mathrm{RWA}}(t)\left(\mathbf{a}^\dagger\mathbf{a}\right)^S - \bar{\Delta}^{\mathrm{RWA}}(t)\mathbf{a}^{\dagger S}\mathbf{a}^S - \frac{\bar{\gamma}^{\mathrm{RWA}}(t)}{2}\left(\mathbf{a}^{\dagger S}\mathbf{a}^\Sigma - \mathbf{a}^S\mathbf{a}^{\dagger\Sigma}\right)\right]\hat{\rho}_S(t). \quad (49)$$

The term proportional to $r^{\mathrm{RWA}}(t)$ gives rise to a renormalization of the oscillator frequency as for the Feynman-Vernon case, described in the previous subsection. Therefore, proceeding with the same considerations and passing to the interaction picture, we obtain the following generalized Master Equation for the system

$$\frac{\mathrm{d}\hat{\rho}}{\partial t} = -\frac{\bar{\Delta}^{\mathrm{RWA}}(t) + \gamma^{\mathrm{RWA}}(t)}{2}\left[\hat{a}^\dagger\hat{a}\hat{\rho} - 2\hat{a}\hat{\rho}\hat{a}^\dagger + \hat{\rho}\hat{a}^\dagger\hat{a}\right] - \frac{\bar{\Delta}^{\mathrm{RWA}}(t) - \gamma^{\mathrm{RWA}}(t)}{2}\left[\hat{a}\hat{a}^\dagger\hat{\rho} - 2\hat{a}^\dagger\hat{\rho}\hat{a} + \hat{\rho}\hat{a}\hat{a}^\dagger\right]. \quad (50)$$

Let us note that this Master Equation, differently from the one obtained for the FV coupling (see Eq. (38)), is in the Lindblad form as far as the time dependent coefficients $\bar{\Delta}^{\mathrm{RWA}}(t) \pm \gamma^{\mathrm{RWA}}(t)$ are positive. This is usually the case for typical reservoir spectra and parameters, as we have discussed in [24].

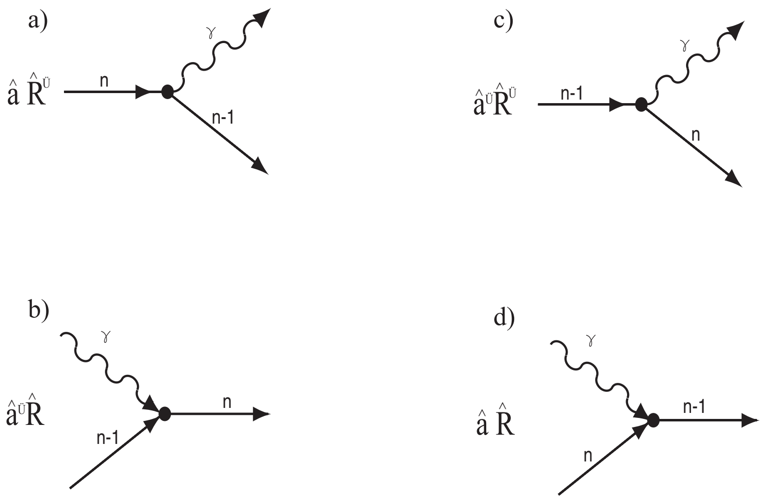

Fig. 1. There are four distinct terms in the Hamiltonian given by equation (52). The events represented in the first two diagrams (a, b) correspond to real processes.The last two diagrams (c, d), instead, describe events corresponding to virtual processes.

4 Comparison between the RW and the Feynman-Vernon coupling models

In the previous section we have seen that starting from a FV coupling or from a RW coupling of an harmonic oscillator with a thermal reservoir it is possible to obtain a generalized Master Equation local in time describing the dynamics of the oscillator. This fact is not surprising. Indeed, as underlined by Paz and Zurek in [21], "perturbative Master Equations can always be shown to be local in time". It is worth noting that, as far as the FV interaction model is concerned, an exact Master Equation, valid for every value of the coupling strength, has been derived [22]. We have verified that the FV Master Equation given by equation (38) coincides with the weak coupling limit of the exact Master Equation of reference [22].

The Master Equations we have derived in the paper are based on the weak coupling assumption but do not rely on the Born-Markov approximation so we are able to examine the non-Markovian short time behavior of the system under study. In addition such equations of course describe the correct Markovian long time asymptotic behavior [24].

The different structure of the two Master Equations given by equation (38) and (50), traceable back to the two different coupling Hamiltonians, are responsible for the occurrence of some physically transparent changes in the oscillator dynamics, more marked in short time regime.

To better understand the physical origin of such differences let us have a closer look at the two interaction Hamiltonians:

$$\hat{H}_{sr}^{\mathrm{RWA}} = \alpha\left(\hat{a}\hat{R}^\dagger + \hat{a}^\dagger\hat{R}\right) \tag{51}$$

$$\hat{H}_{sr}^{\mathrm{FV}} = \alpha\left[\left(\hat{a}\hat{R}^\dagger + \hat{a}^\dagger\hat{R}\right) + \left(\hat{a}\hat{R} + \hat{a}^\dagger\hat{R}^\dagger\right)\right]. \tag{52}$$

We take advantage of a pictorial representation of the four different interaction terms appearing in Hamiltonian (52) (see Fig. 1). The events represented in the first two diagrams (a, b) are processes of absorption or emission in which energy is conserved. The last two diagrams (c, d), on the contrary, describe events not corresponding to real absorption and emission processes. For this reason such processes are called *virtual processes*. In the second order in perturbation theory both the two real and virtual processes combine to give rise to real processes hereafter called alpha and beta processes respectively (see Fig. 2).

Thus when we use the Feynman-Vernon coupling instead of the Rotating Wave one, the channels through which the oscillator exchanges energy with the reservoir are doubled. The asymptotic long time behavior describes, of course, thermalization in both cases. These particular features give rise to different predictions of the short time behavior of physical quantities, such as for example the mean number of quanta $\langle\hat{n}\rangle(t)$ of the system oscillator (heating function), depending on which of the two system-reservoir coupling models is used. We show in the following that such different behaviors are, in principle, experimentally observable and thus relevant for the correct description of the complete dynamics of the system.

Let us consider, as initial state of the system, the vacuum. It is well known that, in this case, due to the interaction with the thermal reservoir at T temperature, the system experiences heating processes leading to thermalization. In reference [24] it has been shown that, for the FV coupling, the non-Markovian time evolution of $\langle\hat{n}\rangle(t)$, in the weak coupling limit, is given by [23,24]

$$\langle\hat{n}\rangle(t \ll \omega_c^{-1}) \rightarrow \left[2\alpha^2 \int_0^\infty \omega|g(\omega)|^2 \left(n(\omega) + \frac{1}{2}\right) \mathrm{d}\omega\right] \frac{t^2}{2}, \tag{53}$$

where $n(\omega)$ is the mean number of reservoir excitations at T temperature and $g(\omega)$ is the reservoir spectral density.

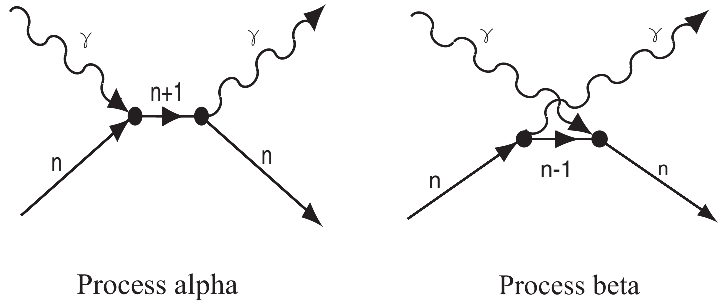

Fig. 2. Two real processes combine to give rise to the real process alpha. Two virtual processes combine to give rise to the real process beta.

For the RW coupling, similar calculations yields the following expression

$$\langle \hat{n} \rangle^{\mathrm{RWA}}(t \ll \omega_c^{-1}) \to \left[\alpha^2 \int_0^\infty \omega |g(\omega)|^2 n(\omega) \mathrm{d}\omega \right] \frac{t^2}{2} \cdot \tag{54}$$

Comparing these last two equations one sees immediately that, for short time intervals, $\langle \hat{n} \rangle(t) \approx 2\langle \hat{n} \rangle^{\mathrm{RWA}}(t)$, meaning that the system-reservoir FV coupling model predicts an initial heating of the system faster than the one predicted by the RW coupling model. This fact can be easily traced back to the doubling of channels for energy exchange illustrated in Figure 1. Note that both equations (53) and (54) show an initial quadratic time dependence. In other words for short times the depopulation of the ground state due to heating is nonexponential. Such a circumstance suggests the possibility of hindering the heating process by performing rapid measurements of the ground state population. However, as the quadratic behavior persists for a short time only, such a technique could be very difficult to implement experimentally [25].

On the other hand, in the long time asymptotic limit, $\langle \hat{n} \rangle(t)$ and $\langle \hat{n} \rangle^{\mathrm{RWA}}(t)$ have, as expected, the same temporal behavior [23,24]:

$$\begin{aligned} \langle n \rangle(\ t \gg \omega_0^{-1}) &= \langle n \rangle^{\mathrm{RWA}}(t \gg \omega_0^{-1}) \\ &\simeq n(\omega_0)(1 - \exp[-\pi\alpha^2\omega_0|g(\omega_0)|^2 t]), \end{aligned} \tag{55}$$

since, due to the time-energy uncertainty principle, for long times t, β processes (see Fig. 2) are very unlikely to happen in the weak coupling regime.

Summing up the two system-reservoir coupling models under scrutiny predict the same asymptotic long time behavior for the observable $\langle \hat{n} \rangle(t)$ but different non-Markovian short time behaviors. It is worth noting that, once known the system and reservoir parameters, the only phenomenologic constant is the coupling constant α. Such quantity is usually estimated from the experiments [26]. If we now assume that experiments may be performed in all the relevant time scale, that is both in the asymptotic long time regime and in the non-Markovian short time regime, one can use the value of the coupling constant experimentally measured in the asymptotic long time regime (see Eq. (55)) to verify if the correct short time behavior is actually the one predicted by equation (53)(FV coupling) or the one given by equation (54) (RW coupling). In fact, one would expect that, since the complete Feynman-Vernon coupling is more general than the RW coupling, it is also more fundamental and thus it should give the correct description of the dynamics of the system.

5 The RWA in the Feynman-Vernon model: comparison with the RW model

Let us now consider again the final form of the generalized Master Equation, given by equation (38) with equations (39–40), derived for the FV coupling. To further simplify the calculation one could think to perform a Rotating Wave Approximation (RWA) averaging on an interval Δt the rapidly oscillating trigonometric functions appearing in equation (38) through equations (39–40). Under such conditions, that is for $2\omega_0 \Delta t \gg 1$, equation (38) assumes the form

$$\begin{aligned} \frac{\mathrm{d}\hat{\rho}(t)}{\mathrm{d}t} = &- \left[\frac{\bar{\Delta}(t)}{2} \left((\mathbf{X}^S)^2 + (\mathbf{P}^S)^2 \right) \right. \\ &\left. + \mathrm{i}\frac{\gamma(t)}{2} \left(\mathbf{X}^S \mathbf{P}^\Sigma - \mathbf{P}^S \mathbf{X}^\Sigma \right) \right] \hat{\rho}(t). \end{aligned} \tag{56}$$

Having in mind equations (3) and (31), after some straightforward calculations the following Master Equation is obtained

$$\begin{aligned} \frac{\mathrm{d}\hat{\rho}}{\partial t} = &- \frac{\bar{\Delta}(t) + \gamma(t)}{2} \left[\hat{a}^\dagger \hat{a} \hat{\rho} - 2\hat{a}\hat{\rho}\hat{a}^\dagger + \hat{\rho}\hat{a}^\dagger \hat{a} \right] \\ &- \frac{\bar{\Delta}(t) - \gamma(t)}{2} \left[\hat{a}\hat{a}^\dagger \hat{\rho} - 2\hat{a}^\dagger \hat{\rho}\hat{a} + \hat{\rho}\hat{a}\hat{a}^\dagger \right], \end{aligned} \tag{57}$$

with $\bar{\Delta}(t)$ and $\gamma(t)$ defined by equations (33–34). It is important to note that equation (57) is in the Lindblad form as far as the sum and difference coefficients ($\bar{\Delta}(t) - \gamma(t)$) and ($\bar{\Delta}(t) + \gamma(t)$) are positive [23].

In reference [24] we have used the Master Equation given by equation (57) to describe the heating process of single trapped ions. Using an operatorial approach we have developed in [23] to solve time-convolutionless Master Equations, we were able to study both the short time non-Markovian and the asymptotic long time Markovian behaviors of the heating function.

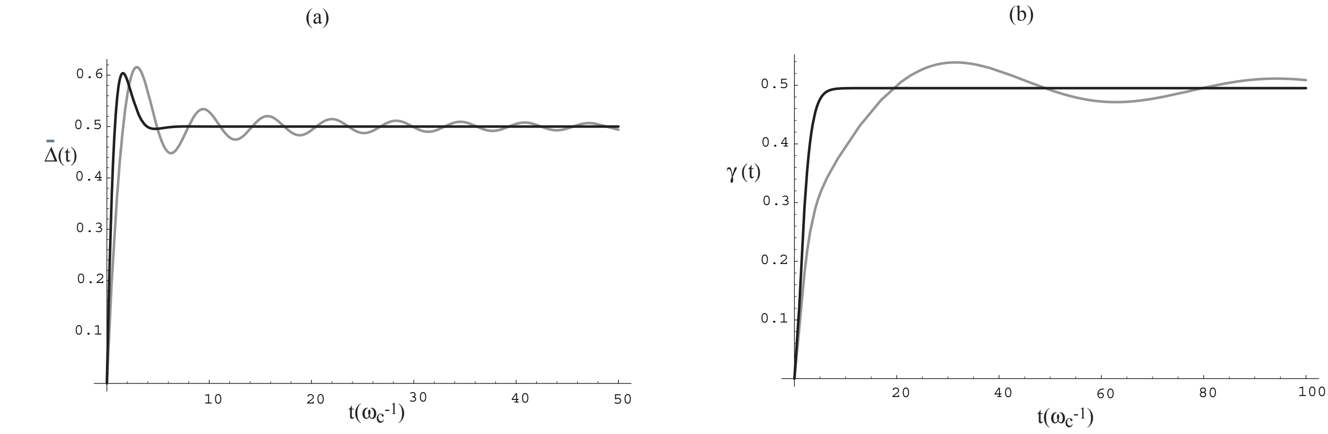

Fig. 3. (a) Asymptotic long time behavior of the coefficients $\bar{\Delta}(t)$ (black line) and $\bar{\Delta}^{\mathrm{RWA}}(t)$ (gray line). (b) Asymptotic long time behavior of the coefficients $\gamma(t)$ (black line) and $\gamma^{\mathrm{RWA}}(t)$ (gray line). In both graphics we have put $\omega_c = \omega_0$.

An interesting feature of equation (57) is that it has the same structure of the Master Equation obtained starting from the RW coupling (see Eq. (50)). Indeed, one sees immediately that the difference between the Master Equation obtained starting form the FV coupling and performing after the RWA, and that one obtained starting from the RW coupling *relies only on the time dependent coefficients of the ME*. Let us have a closer look at the form of such coefficients. In the limit of continuous modes they are written as:

$$\bar{\Delta}(t) = \\ 2\alpha^2 \int_0^t \int_0^\infty \omega |g(\omega)|^2 \left(n(\omega) + \frac{1}{2}\right) \cos[\omega\tau] \cos[\omega_0 \tau] \mathrm{d}\omega \mathrm{d}\tau, \tag{58}$$

$$\bar{\Delta}^{\mathrm{RWA}}(t) = \alpha^2 \int_0^t \int_0^\infty \omega |g(\omega)|^2 \left(n(\omega) + \frac{1}{2}\right) \times \cos[(\omega - \omega_0)\tau] \mathrm{d}\omega \mathrm{d}\tau, \tag{59}$$

$$\gamma(t) = 2\alpha^2 \int_0^t \int_0^\infty \frac{\omega}{2} |g(\omega)|^2 \sin[\omega\tau] \sin[\omega_0\tau] \mathrm{d}\omega \mathrm{d}\tau, \tag{60}$$

$$\gamma^{\mathrm{RWA}}(t) = \alpha^2 \int_0^t \int_0^\infty \frac{\omega}{2} |g(\omega)|^2 \cos[(\omega - \omega_0)\tau] \mathrm{d}\omega \mathrm{d}\tau, \tag{61}$$

where $n(\omega) = \left(\exp[\frac{\hbar\omega}{kT}] - 1\right)^{-1}$ is the number of reservoir excitations at T temperature. In the following we assume an Ohmic environment characterized by a reservoir spectral density having frequency cut-off ω_c, as for example the Drude spectral density

$$|g(\omega)|^2 = \frac{1}{\pi} \frac{\omega_c^2}{\omega_c^2 + \omega^2} \,. \tag{62}$$

A noticeable difference between the $\bar{\Delta}(t)(\gamma(t))$ and $\bar{\Delta}^{\mathrm{RWA}}(t)(\gamma^{\mathrm{RWA}}(t))$ coefficients is that in the last one the anti-resonant term $\cos[(\omega + \omega_0)\tau]$ is absent. Such a circumstance leads to distinguishable short time behaviors of the FV and RW coefficients.

It is indeed possible to prove that in $\bar{\Delta}(t)$, for $t \ll \omega_c^{-1}$, alpha and beta processes give rise to the same contributions linear in t so that $\bar{\Delta}(t) \approx 2\bar{\Delta}^{\mathrm{RWA}}(t)$.

As far as $\gamma(t)$ is concerned, on the contrary, the same processes cancel each other at the first order in t in such a way that, for $t \ll \omega_c^{-1}$, $\gamma(t) \propto t^3$ whereas $\gamma^{\mathrm{RWA}}(t) \propto t$.

In the asymptotic Markovian long time regime we have, as expected, that $\bar{\Delta}(t \gg \omega_c^{-1}) \simeq \bar{\Delta}^{\mathrm{RWA}}(t \gg \omega_c^{-1})$ and $\gamma(t \gg \omega_c^{-1}) \simeq \gamma^{\mathrm{RWA}}(t \gg \omega_c^{-1})$, as shown in Figure 3

At this point it is worth making some considerations on the validity of the RWA performed to derive equation (57). As we have already said at the beginning of this section, the RWA consists in neglecting terms oscillating at the frequency $2\omega_0$. In other words performing the RWA amounts at looking at the course-grained structure of the dynamics of the systems. For this reason we cannot describe correctly the dynamical features in a time interval such as $\Delta t \leq \omega_0^{-1}$. Very often one deals with situations in which the characteristic frequency of the system ω_0 is smaller or much smaller than the reservoir frequency cut ω_c. Under this circumstances, normally, we cannot rely on the short time expressions of the FV coefficients $\bar{\Delta}(t)$ and $\gamma(t)$ since they are valid for times $t \ll \omega_c^{-1} \ll \omega_0^{-1}$. However, there are two cases in which one can use the Master Equation given by equation (57) to describe correctly the non-Markovian short time behavior of the system:

1. whenever one wants to look at situations in which $\omega_0 > \omega_c$, as discussed for example in [16,25];
2. whenever we are interested in the mean value of a certain class of observables, like for instance the number operator $\langle \hat{a}^\dagger \hat{a} \rangle(t)$ (see [23,24]).

In this last case, indeed, it has been shown [23] that, in the weak coupling limit, it is equivalent to use the solution of the the Master Equations (38) or (57) since they lead to the the same analytic expressions for the expectation value of the observable of the class before mentioned.

6 Summary and conclusions

We have described a procedure, based on superoperator formalism, to derive, in the weak coupling limit, non-Markovian generalized Master Equations local in time. Such a method is equivalent to the time-convolutionless projection operator technique in the sense that it leads to the same generalized Master Equation. We apply this procedure to derive the Master Equation for a specific system, namely a quantum harmonic oscillator coupled to a thermal reservoir at T temperature. We compare two different microscopic system-reservoir coupling models: the Feynman-Vernon and the rotating wave couplings. Both couplings are bilinear, but the first one is more general and thus, in this sense, it is more fundamental. Very often however, in quantum optics systems, the Rotating Wave coupling is used because the counter rotating terms not conserving the unperturbed energy cannot contribute to the system dynamics. The main result of our paper is to establish under which conditions such a claim is effectively correct. By comparing the analytic solutions of the heating function relative to the two different coupling models (FV and RW couplings) we conclude that, even in the weak coupling limit, the counter rotating terms give indeed a significant contribution in the non-Markovian short time regime. Such a contribution is actually experimentally measurable, provided that one can perform experiments in all the time scale relevant for the system dynamics. To this purpose it is worth noting that in the context of trapped ions experiments have been performed in which the system (single harmonic oscillator) is first cooled down to its zero point energy and then coupled to a properly engineered reservoir [26]. We note that, in such experiments, it is possible not only to choose at will the reservoir parameters, but also to engineer the coupling and control the coupling strength. Therefore, the great experimental advances of the trapped ion techniques could make it possible to perform an experiment aimed at proving the relevant role, in the short time dynamics, of the usually neglected counter rotating terms.

One of the reasons for which one usually prefers to work with master equations derived starting from the RW coupling model is related to the fact that the resulting Master Equation, in the Born-Markov approximation, is in the Lindblad form differently from the case in which the Feynman-Vernon coupling is assumed (see Master Equation for Brownian motion). We have demonstrated here that also the non-Markovian Master Equation obtained starting from the RW coupling is in the Lindblad form, for some value of the relevant system and reservoir parameter. Moreover, by looking at the analytic expression of the time dependent coefficients of our non-Markovian generalized Master Equations one can infer the conditions under which one passes from Lindblad to non Lindblad Master Equations. Remembering that the Master Equations given by equations (50) and (57) are of Lindblad type when their time dependent coefficients are positive, indeed, it is not difficult to convince oneself that such conditions are simply related to the change of the sign of the coefficients. Therefore the form of the RW Master Equations derived in this paper allows to study the border separating two very different physical regimes characterized by very different system dynamics [16] and, for this reason, makes it possible to gain more insight in the fundamental dissipative processes of one of the most extensively studied physical systems: a harmonic oscillator coupled to a thermal reservoir.

Another new result we have obtained in this paper stems from the comparison between the master equations derived in the following two cases:

1) Feynman-Vernon system reservoir coupling followed by the RWA performed after tracing over the reservoir degrees of freedom;

2) Rotating Wave system reservoir coupling.

Stated another way we look at the differences in the system dynamics arising from the two following approximations respectively:

1) average over rapidly oscillating terms after tracing over the reservoir variables;

2) neglecting the counter rotating terms in the initial microscopic coupling model.

We have shown that the Master Equation obtained from the Feynman-Vernon coupling, after performing the RWA, is of the Lindblad type and it actually has the same structure of the RW Master Equation, with different time dependent coefficients. We have demonstrated that these two different approximations lead to different short time behaviors, while in the asymptotic long time Markovian regime the two correspondent Master Equations do coincide. However we have proved that performing the RWA after tracing over the reservoir variables is a less restrictive approximation than starting with the RW coupling model. Indeed, differently from the RW Master Equation, the Feynman-Vernon one + RWA, takes into account the virtual photon exchanges relevant in the short time dynamics and thus it predicts the correct non-Markovian short time behavior, provided that $t \gg \omega_0^{-1}$.

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Appendix A: The general Master Equation

In this Appendix we sketch the derivation of equation (21) from equation (20). Let us consider an interaction Hamiltonian with the form

$$\hat{H}_I(t) = \alpha \hat{A}_i(t)\hat{E}_i(t) = \alpha \hat{\boldsymbol{A}}(t) \cdot \hat{\boldsymbol{E}}(t), \qquad (63)$$

where for simplicity we have used Einstein notation. Using some algebraic properties of the superoperators one can show that if $[\hat{A}_i, \hat{E}_i] = 0$ then

$$(\hat{A}_i(t)\hat{E}_i(t))^S = \frac{1}{2}\left(\mathbf{A_i}^S\mathbf{E_i}^\Sigma + \mathbf{A_i}^\Sigma\mathbf{E_i}^S\right). \qquad (64)$$

Exploiting the properties of the trace and the assumption $\langle \hat{E}_i \rangle = \mathrm{tr}\hat{E}_i\hat{\rho}_E E = 0$, one gets

$$\mathrm{tr}\mathbf{E}_i^S\hat{\rho}_E(0)E \equiv 0, \qquad \mathrm{tr}\mathbf{E}_i^\Sigma\hat{\rho}_E(0)E = 0. \qquad (65)$$

Consequently

$$\langle \mathbf{H_I}^S(t)\rangle = \mathrm{tr}\mathbf{E}_i^S\hat{\rho}_E(0)E = \frac{1}{2}\left(\langle \mathbf{E}_i^\Sigma(t)\rangle \mathbf{A}_i^S(t) + \langle \mathbf{E}_i^S(t)\rangle \mathbf{A}_i^\Sigma(t)\right) = 0. \quad (66)$$

In the same manner it is not difficult to show that

$$\langle \mathbf{H_I}^S(t)\mathbf{H_I}^S(t_1)\rangle = \frac{1}{4}\left(\langle \mathbf{E}_i^\Sigma(t)\mathbf{E}_j^\Sigma(t_1)\rangle \mathbf{A}_i^S(t)\mathbf{A}_j^S(t_1) + \langle \mathbf{E}_i^\Sigma(t)\mathbf{E}_j^S(t_1)\rangle \mathbf{A}_i^S(t)\mathbf{A}_j^\Sigma(t_1)\right), \quad (67)$$

where the equalities

$$\langle \mathbf{E}_i^S(t)\mathbf{E}_j^\Sigma(t_1)\rangle = \langle \mathbf{E}_i^S(t)\mathbf{E}_j^S(t_1)\rangle \equiv 0, \quad (68)$$

have been used. After some algebraic manipulation one gets

$$\langle \mathbf{E}_i^\Sigma(t)\mathbf{E}_j^\Sigma(t_1)\rangle = 2\langle\{\hat{E}_i^\Sigma(t-t_1), \hat{E}_j^\Sigma(0)\}\rangle, \quad (69)$$

$$\langle \mathbf{E}_i^\Sigma(t)\mathbf{E}_j^S(t_1)\rangle = 2\langle[\hat{E}_i^\Sigma(t-t_1), \hat{E}_j^\Sigma(0)]\rangle, \quad (70)$$

where the square an curl brackets indicate the commutator and anti-commutator respectively.

Substituting these expressions into equation (20) and using the definitions of the correlation and susceptibility matrices one obtains the general Master Equation given by (21).

Appendix B: Derivation of FV Master Equation

In this appendix we present the superoperatorial mathematical properties allowing to derive the final form of the FV Master Equation, given by equation (38) discussed in this paper. First of all let us consider the following superoperatorial relations

$$[\mathbf{A}^S, \mathbf{B}^S] = [\mathbf{A}^\Sigma, \mathbf{B}^\Sigma] = [\hat{A}, \hat{B}]^S, \quad (71)$$

$$[\mathbf{A}^S, \mathbf{B}^\Sigma] = [\hat{A}, \hat{B}]^\Sigma. \quad (72)$$

Starting from these equations and having in mind the Baker-Hausdorff formula one gets

$$\mathbf{A}^{S(\Sigma)}(t) \equiv \exp\left[\mathrm{i}\mathbf{B}^S t\right]\ \mathbf{A}^{S(\Sigma)}\ \exp\left[-\mathrm{i}\mathbf{B}^S t\right] = \left(\exp\left[\mathrm{i}\hat{B}^S t\right]\ \hat{A}\ \exp\left[-\mathrm{i}\hat{B}^S t\right]\right)^{S(\Sigma)} = \left(\hat{A}(t)\right)^{S(\Sigma)}. \quad (73)$$

The previous relation says that the time evolution of superoperators is ruled by equations analogue to those of the operators in Dirac's formalism. Then, specifying equation (21) to the system under scrutiny and using equations (24) and (28) yields

$$\mathbf{D}_S(t) = \int_0^t \kappa(\tau)\mathbf{T}_0(t)\ \mathbf{X}^S(t)\mathbf{X}^S(t-\tau)\ \mathbf{T}_0^{-1}(t)\mathrm{d}\tau = \int_0^t \kappa(\tau)\mathbf{X}^S\mathbf{X}^S(-\tau)\mathrm{d}\tau = \int_0^t \kappa(\tau)\mathbf{X}^S\left(\cos\omega_0\tau\mathbf{X}^S - \sin\omega_0\tau\mathbf{P}^S\right)d\tau, \quad (74)$$

$$\mathbf{G}_S(t) = \int_0^t \mu(\tau)\mathbf{T}_0(t)\ \mathbf{X}^S(t)\mathbf{X}^\Sigma(t-\tau)\ \mathbf{T}_0^{-1}(t)\mathrm{d}\tau = \int_0^t \mu(\tau)\mathbf{X}^S\mathbf{X}^\Sigma(-\tau)\mathrm{d}\tau = \int_0^t \mu(\tau)\mathbf{X}^S\left(\cos\omega_0\tau\mathbf{X}^\Sigma - \sin\omega_0\tau\mathbf{P}^\Sigma\right)\mathrm{d}\tau. \quad (75)$$

Appendix C: Derivation of the RW Master Equation

In this Appendix we underline the essential steps in the derivation of the RW Master Equation given by equation (41). Let us note that, in interaction picture, the Hamiltonian given by equation (1) assumes the form

$$\hat{H}_{sr}^{\mathrm{RWA}}(t) = \alpha\sum_{i=0}^{\infty}\hbar\sqrt{\frac{\omega}{2}}\left(g_i\hat{a}\mathrm{e}^{-\mathrm{i}\omega_0 t}\hat{b}_i^\dagger\mathrm{e}^{\mathrm{i}\omega_i t} + \mathrm{h.c.}\right). \quad (76)$$

From a mathematical point of view it is convenient to associate all the time dependent phase factors to the reservoir operators as follows

$$\hat{H}_{sr}^{\mathrm{RWA}}(t) \equiv \alpha\left(\hat{a}\hat{\tilde{R}}^\dagger(t) + \hat{a}^\dagger\hat{\tilde{R}}(t)\right), \quad (77)$$

with

$$\hat{\tilde{R}}(t) \equiv \sum_{i=0}^{\infty}\hbar\sqrt{\frac{\omega}{2}}g_i\hat{b}_i\mathrm{e}^{-\mathrm{i}(\omega_i-\omega_0)t}. \quad (78)$$

For the system here considered, thus, the operators appearing in the bilinear form defined in equation (19) are $\hat{\boldsymbol{A}}(t) = \hat{\boldsymbol{A}} = (\hat{a}, \hat{a}^\dagger)$ and $\hat{\boldsymbol{E}}(t) = (\hat{\tilde{R}}^\dagger(t), \hat{\tilde{R}}(t))$.

Exploiting the properties of superoperators given by equation (71) and (72), with some algebraic manipulation, the Master Equation given by equation (21) can be recast in the form given by equation (41). Finally we write such a Master Equation in the Schrödinger picture exploiting of the following property

$$\mathbf{T}_0(t)\ \mathbf{a}^{\dagger S}\mathbf{a}^{S(\Sigma)}\ \mathbf{T}_0^{-1}(t) = (\hat{a}^\dagger\ \mathrm{e}^{-\mathrm{i}\omega_0 t})^S(\hat{a}\ \mathrm{e}^{\mathrm{i}\omega_0 t})^{S(\Sigma)} = \mathbf{a}^{\dagger S}\mathbf{a}^{S(\Sigma)}, \quad (79)$$

with $\mathbf{T}_0(t)$ defined by equation (26). Concluding, we note that, the superoperator proportional to $r^{\mathrm{RWA}}(t)$, appearing in equation (21), can be recast in the form

$$\frac{1}{2}\left(\mathbf{a}^{\dagger S}\mathbf{a}^\Sigma + \mathbf{a}^S\mathbf{a}^{\dagger\Sigma}\right) = \left(\hat{a}^\dagger\hat{a}\right)^S. \quad (80)$$

Thus the corresponding term in the RW Master Equation is a frequency renormalization term. Neglecting this term and going back to the interaction picture one gets the final form of the Master Equation, given by equation (50).

References

1. C.W. Gardiner, *Quantum Noise* (Springer-Verlag, Berlin, 1991); H.J. Carmichael, *An Open System Approach to Quantum Optics* (Springer-Verlag, Berlin, 1993)
2. W.H. Zurek, Phys. Today **44**, 36 (1991)
3. W.H. Zurek, Phys. Rev. D **24**, 1516 (1981)
4. W.H. Zurek, Phys. Rev. D **26**, 1862 (1982)
5. D. Ahn, J. Lee, M.S. Kim, S.W. Wang, Phys. Rev. A, **66**, 012302 (2002)
6. B. Misra, E.C.G. Sudarshan, J. Math. Phys. **18**, 7456 (1977); P. Facchi, S. Pascazio, *Progress in Optics*, 41, edited by E. Wolf (Elsevier, Amsterdam, 2001)
7. S. Chaturvedi, J. Shibata, Z. Phys. B **35**, 297 (1979); N.H.F. Shibata, Y. Tagahashi, J. Stat. Phys. **17**, 171 (1977)
8. H.P. Breuer, F. Petruccione, *The Theory of Open Quantum systems* (Oxford University Press, 2002)
9. G.W. Ford, M. Kac, P. Mazur, J. Math. Phys. **6**, 504 (1965)
10. G.W. Ford, M. Kac, J. Stat. Phys. **6**, 803 (1987)
11. A.O. Caldeira, A.J. Legget, Ann. Phys. **149**, 374 (1983)
12. R.P. Feynman, F.L. Vernon, Ann. Phys. (N.Y.) **24**, 118 (1963)
13. C. Cohen-Tannoudji, J. Dupont-Roc, G. Grynberg, *Atom-Photon Interaction* (John Wiley and Son, New York, 1992)
14. G.M. Moy, J.J. Hope, C.M. Savage, Phys. Rev. A **59**, 667 (1999); J.J. Hope, Phys. Rev. A, **55**, R2531 (1997)
15. A.O. Caldeira, A.J. Legget, Physica A **121**, 587 (1983); B.L. Hu, J.P. Paz, Y.H. Zhang, Phys. Rev. D **45**, 2843 (1992); B.L. Hu, J.P. Paz, Y.H. Zhang, Phys. Rev. D **47**, 1576 (1993)
16. H.-P. Breuer, B. Kappler, F. Petruccione, Phys. Rev. A **59**, 1633 (1999).
17. N.G. Van Kampen, Physica **74**, 215 (1974)
18. N.G. Van Kampen, Physica **74**, 239 (1974)
19. R. Kubo, J. Phys. Soc. Jpn **17**, (1962) 1100
20. N.G. Van Kampen, *Stochastic Processes in Physics and Chemistry* (NH PL, Amsterdam 1985)
21. J.P. Paz, W.H. Zurek, in *Proceedings of the 72nd Les Houches summer school: Coherent atomic matter waves*, edited by R. Kaiser, C. Westbrook, F. David (EDP Sciences and Springer-Verlag, 1999)
22. B.L. Hu, J.P. Paz, Y. Zhang, Phys. Rev. D **45**, 2843 (1992)
23. F. Intravaia, S. Maniscalco, A. Messina, *Density matrix operatorial solution of the non-Markovian Master Equation for Quantum Brownian Motion*, `quant-ph/0211196`
24. F. Intravaia, S. Maniscalco, J. Piilo, A. Messina, Phys. Lett. A **308**, 6 (2003)
25. D.F.V. James, Phys. Rev. Lett. **81**, 317 (1998)
26. C.J. Myatt *et al.*, Nature **403**, 269 (2000); Q.A. Turchette *et al.*, Phys. Rev. A **62**, 053807 (2000)